三角関数の積分を有理関数の積分に変換する一般的方法

$\tan\frac{x}{2}=t$ とおくと，

$$\tan x=\frac{2t}{1-t^2},\quad \sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2},\qquad dx=\frac{2}{1+t^2}dt$$

注意

却って計算が煩雑になることもある。おそらく入試問題はそう作ってあるでしょう。

解説

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{2t}{1-t^2}$$

$$\sin x=\frac{\sin x}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}{\dfrac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{2\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\left(\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2}=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos x}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{\dfrac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{\dfrac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}=\frac{1-\left(\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2}{1+\left(\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2}=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$$\frac{d}{dx}\tan\frac{x}{2}=\frac{dt}{dx}\qquad ・・・①$$

$$\frac{d}{dx}\tan\frac{x}{2}=\frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}}=\frac{1}{2}\cdot\frac{1}{\dfrac{1}{1+\tan^2\frac{x}{2}}}=\frac{1+\tan^2\frac{x}{2}}{2}=\frac{1+t^2}{2}\qquad ・・・②$$

①，②より，$\dfrac{1+t^2}{2}=\dfrac{dt}{dx}$

よって，$dx=\dfrac{2}{1+t^2}dt$

例 1

$$
\begin{aligned}
\int \frac{1}{1-\sin x}dx &= \int \frac{1}{1-\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}dt \\
&= \int \frac{2}{(1-t)^2}dt \\
&= \frac{2}{1-t}+C \\
&= \frac{2}{1-\tan\frac{x}{2}}+C \\
&= -\frac{2}{\tan\frac{x}{2}-1}+C
\end{aligned}
$$

例 2

$$
\begin{aligned}
\int \frac{1}{\cos x}dx &= \int \frac{1}{\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt \\
&= \int \frac{2}{1-t^2}dt \\
&= \int \left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt \\
&= -\log|1-t|+\log|1+t|+C \\
&= \log\left|\frac{1+t}{1-t}\right|+C \\
&= \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|+C
\end{aligned}
$$

例 3

$$
\begin{aligned}
\int \frac{1}{1+\sin x+\cos x}dx &= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}dt \\
&= \int \frac{1}{1+t}dt \\
&= \log|1+t|+C \\
&= \log\left|1+\tan\frac{x}{2}\right|+C
\end{aligned}
$$

例 **4**

$$
\begin{aligned}
\int \frac{1}{(1+\cos x)^2}dx &= \int \frac{1}{\left(1+\dfrac{1-t^2}{1+t^2}\right)^2}\cdot\frac{2}{1+t^2}dt \\
&= \int \frac{(1+t^2)^2}{4}\cdot\frac{2}{1+t^2}dt \\
&= \frac{1}{2}\int (1+t^2)dt \\
&= \frac{1}{2}t+\frac{1}{6}t^3+C \\
&= \frac{1}{2}\tan\frac{x}{2}+\frac{1}{6}\tan^3\frac{x}{2}
\end{aligned}
$$